

Condensation of elastic energy in two-dimensional packing of wires

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Forced packing of a long metallic wire injected into a two-dimensional cavity leads to crushed structures involving a hierarchical cascade of loops with varying curvature radii. We study the distribution of elastic energy stored in such systems from experiments, and high-resolution digital techniques. It is found that the set where the elastic energy of curvature is concentrated has dimension $D_S=1.0\pm 0.1$, while the set where the mass is distributed has dimension $D=1.9\pm 0.1$.

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I. INTRODUCTION

Packing problems are of noteworthy importance to many branches of industry and science. Theoretical, experimental, and technological investigations of these problems have attracted much attention in connection with number theory, coding and group theory, analog-to-digital conversion and data compression, d -dimensional crystallography and condensed-matter physics in general, as well as in dual theory and superstring theory [1]. Frequently we are interested in the strategy to obtain the most efficient way to pack a large number of equal extended units, say spheres, in d -dimensional Euclidean space. Even in the area of two-dimensional circle packing, there is a myriad of open important problems of theoretical and applied interest [2]. A less studied problem in two-dimensional packing, which has great technological interest, is the search of a rigid system with a low packing fraction. An example of this last class of systems is a single layer structure of loops obtained when a crushed wire is confined in a two-dimensional cavity [3]. This specific case yields rigid metallic structures with a maximum average occupation fraction of 14% of the area irrespective of the packing [4]. In fact, this maximum packing fraction cannot vary appreciably with the material, because it is controlled by the approximately universal ratio of the shear modulus to the modulus of elasticity of the wire [5].

On the other hand, many physical systems present the phenomenon of condensation of energy, that is, a D -dimensional system in a space of dimension $d\geq D$ can have some type of energy concentrated in a small fraction of its total volume. In principle, the subset \mathcal{S} where this particular energy is localized can have a fractal dimension $D_S\leq D$. At this point it is opportune to introduce the concept of support. In mathematics, the support of a quantity is the set \mathcal{S} where that quantity is nonzero. Many soft-condensed-matter systems as proteins [6], polymers in general [7], and diffusion limited aggregates [8] have in physical space fractal structures with dimension $D<3$. They are formed from a large number of units glued together by short-range covalent chemical forces and/or van der Waals forces. The existence of these systems themselves is a proof of condensation of chemical and van der Waals energies on low dimensional supports. In these cases, the distribution of short-range forces follows the respective structure of the systems, and as a consequence D_S is equal to D .

Condensation of elastic energy, in particular, has raised recently a growing interest in connection with crumpled sheets, and m -dimensional elastic manifolds [9,10]. Other problems related to the physics of crumpled structures with the topology of a sheet have been investigated in the last two decades, as examples one can mention studies on basic statistical aspects [11–15], on stress and strain relaxation in crumpled thin sheets [16], among others [17]. On the contrary, the physics of crushed structures with one-dimensional topology as exemplified by a squeezed ball of wire has been much less studied. Geometrical, statistical, and physical aspects of crushed wires in three-dimensional space were previously examined experimentally from the point of view of robust scaling laws. Fractal dimensions associated with these disordered systems were reported [18].

In this paper we study the condensation of elastic energy in configurations of a copper wire injected into a planar two-dimensional cavity, using experiments, scaling, and high-resolution digital techniques for data acquisition. The structures studied here are remarkably different from crumpled sheets or crushed wires in three dimensions.

II. EXPERIMENTAL DETAILS

Our experiments of packing of wire were performed in a two-dimensional transparent cell consisting of the superposition of two disks of plexiglass with a total height of 1.8 cm, and an external radius of 15 cm. The internal circular cavity of the cell had a radius $R_0=10.0$ cm, and 0.11 cm of height, and it can only accommodate configurations of a single layer of the folded structure of the wire. The cavity of the cell was polished to reduce the friction between wire and cavity. The no. 19AWG copper wire used in the experiments had a diameter $\zeta=0.10$ cm and a varnished surface, in order to reduce wire-wire friction. Cavity and wire operated in the dry regime, free of any lubricant, and the injection of wire into the cell was made through two channels along the diameter as shown in Fig. 1. The photographs were taken with a digital camera with a resolution of 3.2 Mpixel. The digital images were transferred to a personal computer for processing. This stage consists of digital filtering, data validation, and conversion into binary images.

Each packing experiment begins fitting a straight wire in the channels and subsequently pushing manually and uniformly the wire on both sides of the cell toward the interior



FIG. 1. A real packing configuration of no. 19AWG copper wire within the cavity of 20.0 cm of diameter.

of the cavity with a velocity of order of 1 cm/s. However, the observed phenomena are widely independent of the injection speed for the whole interval of injection velocity compatible with a manual process. The packing of an elastic wire in a cavity involves a hierarchical cascade of loops heterogeneously distributed in space as exemplified in Fig. 1. In this figure the packing fraction is $p=0.14$, corresponding to a length $L=440$ cm of wire. This fraction p is defined as the ratio of the projected area of the crushed wire to the area of the cavity and is equal to $\zeta L / \pi R_0^2$. When the length of wire within the cavity increases in the interval $p > 0.10$, the difficulty in the injection begins to rise, with a corresponding reduction in the velocity of injection. For p near the maximum packing fraction p_{max} , the difficulty in the injection rises abruptly and the crushed structures finally become rigid: for $p > p_{max}$, the crushed wire becomes completely jammed within the cavity and it is impossible to continue the injection of wire. The particular moment when the injection velocity goes rapidly to zero leads to a tight-packing (TP) configuration for the crushed wire (Fig. 1). The average maximum packing fraction in the experiment is $p_{max} = 0.14 \pm 0.02$, corresponding to an aspect ratio $L/\zeta = (\pi R_0^2 p / \zeta^2) = (4.40 \pm 0.63) \times 10^3$, irrespective the existence of lubrication in the cavity. The reader can observe that the sharp creases and ridges found in crumpling of sheets are absent in the two-dimensional packing of wire shown in Fig. 1. In order to better visualize the packing process considered here, we refer to our previous work [3,4].

The packing of a copper wire in our experiments is not perfectly reversible [3,4], however, the degree of irreversibility is small, and considerable elastic energy remains stored in the cavity, as we confirm from the observation of the strong uncoiling of the wire when the cell is disconnected after a very long period. As we can see from Fig. 1, the loops are units that have a bulge in one extremity, and in the other extremity the two branches of the loop merge. The bulge is formed by different contiguous small arcs characterized by different radii of curvature ρ , where the elastic energy of curvature, proportional to $(1/\rho)^2$ [10], can be stored in different degrees. The extremity of each loop opposite to the bulge is associated with arcs of very large curvature radii and, as a consequence, its capacity to store elastic energy is comparatively very small. In this paper we concentrate in this particular type of elastic energy, since the energy cost for

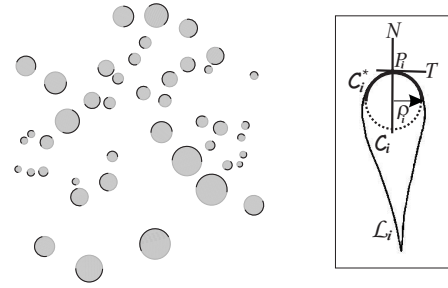


FIG. 2. Support \mathcal{S} of condensation of elastic energy for the packing in Fig. 1. Arcs are regions of the folded structure of wire that contribute to \mathcal{S} . Shading denotes osculatory circles. Inset: loop \mathcal{L}_i with the quantities defined in the text.

stretching the wire is much larger than the bending cost.

Now we define, in the context of this work, the support \mathcal{S} where the elastic energy of the crushed wire is distributed (Fig. 2, inset): for each loop \mathcal{L}_i , we found the point $P_i \in \mathcal{L}_i$ where the loop presents the smaller radius of curvature, ρ_i . Due to the geometry of the loops, this point P_i is localized in the bulge of the loop. The tangent T and its corresponding normal N at P_i are found, and the associated osculatory circle C_i is inserted, as indicated in Fig. 2. If ρ_i is smaller than an arbitrary threshold ρ_c , the arc C_i^* defined by the half part of C_i (one-quarter of the circumference from each side of the normal, with the midpoint of C_i^* at P_i) is considered an element of \mathcal{S} . If, on the contrary, $\rho_i > \rho_c$ we consider that \mathcal{L}_i does not contribute to the support \mathcal{S} . Then, \mathcal{S} is a collection $\{C_i^*\}$ of semicircles, as shown in Fig. 2. This figure refers to the support of energy associated with the crushed wire shown in Fig. 1, and the threshold adopted in this case was $\rho_c = 22\zeta = 2.2$ cm.

III. RESULTS AND DISCUSSION

For testing scale invariance in the support \mathcal{S} , we apply both the box counting and the average mass-length methods [19]. Box counting was performed by overlapping a grid of size ϵ over the set and counting the number $N(\epsilon)$ of squares of size ϵ needed to cover \mathcal{S} . For fractal objects $N(\epsilon) \sim \epsilon^{-D_S}$, where D_S is the fractal dimension. The mass-length relation measures the dependence between the mass $M(R)$ of \mathcal{S} within circles of radii R fully included in the cavity after averaging on many centers and all configurations. In our case, $M(R)$ is given by the number of pixels within the circles. For a scale-free set, it is expected a scaling relation of the type $M(R) \sim R^{D'_S}$, where D'_S is the mass exponent. The main plot in Fig. 3 illustrates the results obtained with the box-covering method; in this case we get $N(\epsilon) \sim \epsilon^{-D_S}$, with $D_S = 1.0 \pm 0.1$ (average on the five experimental configurations). On the other hand, from the power law scaling for $M(R)$ shown in the upper inset of the figure we find $D'_S = 1.0 \pm 0.1$ in agreement with the box counting exponent D_S . These exponents, however, are markedly different from the mass exponent $D = 1.9 \pm 0.1$, found for the entire mass distribution of wire within the cavity in the situation of maximum packing fraction [3,4]. These tests suggest that the support

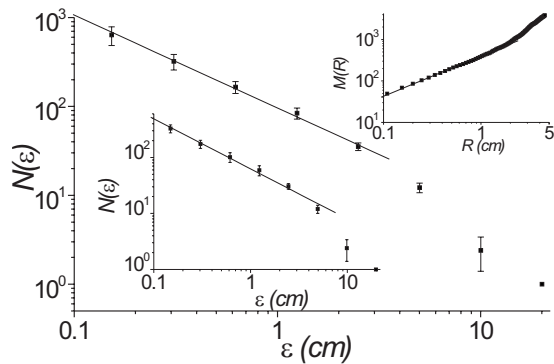


FIG. 3. Main plot: number $N(\epsilon)$ of boxes of size ϵ needed to cover the energy support \mathcal{S} as a function of ϵ . Upper inset: the corresponding mass-length plot. Lower inset: $N(\epsilon)$ when the arcs of \mathcal{S} have angular aperture of 90° instead of 180° . All plots are averages on five experiments.

where energy is condensed, according to our previous definition, has a low dimension, which seems to be a well-defined property of the system. The values of D_S and D'_S are, within the error bars, independent of both ρ_c and the fraction ϕ of the perimeter of each oscillatory circle \mathcal{C}_i included in the support \mathcal{S} (in Fig. 2, we have $\phi=0.5$, corresponding to an angular aperture of 180°). To illustrate this last aspect, we give in the lower inset of Fig. 3 the plot of $N(\epsilon)$ for the energetic arcs when the angular aperture of the arcs is reduced to 90° ($\phi=0.25$); in this case, $N(\epsilon) \sim \epsilon^{-D_S}$, with $D_S = 0.9 \pm 0.1$.

Finally we present an analysis of how the wire-wire contacts are geometrically distributed across the cavity. In the packing of wires, as we can observe from Fig. 1, there are tangency regions, i.e., regions presenting n -fold wire-wire contacts. In an n -fold contact, $n+1$ pieces of wire come together into direct contact as detailed in Fig. 4(a). Each contact region has an area given by the length of the tangency region multiplied by the width $(n+1)\zeta$ of the corresponding number $n+1$ of segments of wire contributing to the contact. The plot in Fig. 4(b) shows the result of a box counting analysis for the patterns of wire-wire contacts for the ensemble of configurations of crushed wire examined in this paper. The experimental data are well described by a power law decay suggesting a fractal dimension $D_{ww} = 1.2 \pm 0.1$, a value close to D_S and D'_S . We speculate that the wire-wire contacts define arcs of propagating forces across the structure as suggested by the five dashed lines in the lower inset of Fig. 4(c). The impossibility to eliminate completely the friction along wire-wire interfaces leads to dashed lines only approximately orthogonal to the contact regions. The nu-

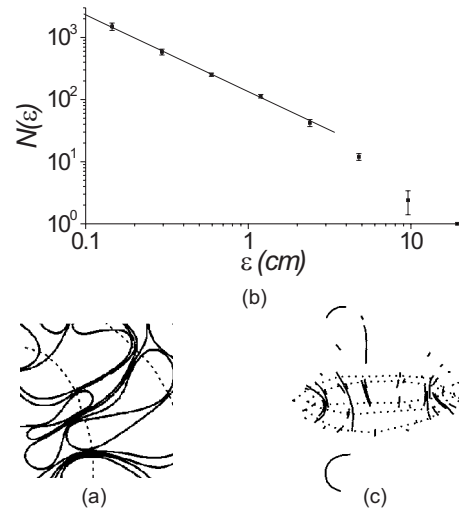


FIG. 4. (a) Detail of loops and wire-wire contacts. (b) Box counting function $N(\epsilon)$ for wire-wire contacts for the ensemble of crushed wires. (c) Typical full pattern of wire-wire contacts. Dashed lines connecting wire-wire contacts are possibly paths for propagation of forces across the structure.

merical value obtained for D_{ww} seems in agreement with the idea that forces are transmitted primarily along one-dimensional structures in noncrystalline systems as, e.g., observed in granular materials [20–22].

IV. CONCLUSION

In summary, we have studied experimentally in detail the geometry of the support where the elastic energy of curvature is stored in a packing of crushed wire in a two-dimensional cavity. Robust scaling laws connecting variables of interest are reported, and the associated critical exponents are determined. Based on the experimental data for two-dimensional systems, it is found that the elastic energy is concentrated on a set of dimensionality close to unity. It is possible that in a d -dimensional cavity, the elastic energy could be condensed equally on a low-dimensional support with dimension $D_S \cong 1$. Elastic materials such as steel wires and nylon fishing lines exhibit patterns of folds as that shown in Fig. 1. This indicates that D_S in these cases has the same value reported in this paper.

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